

AN EXPLICIT COMPUTATION OF A FAMILY OF TRIVIALISING ÉTALE COVERS

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ABSTRACT. We explicitly compute étale covers of the smooth Fermat curves $Y_{p+1} = \text{Proj } k[u, v, w]/(u^{p+1} + v^{p+1} - w^{p+1})$ which trivialise the vector bundles $\text{Syz}(u^2, v^2, w^2)(3)$, where k is a field of characteristic $p \geq 3$.

INTRODUCTION

In this paper we explicitly compute a trivialising étale cover $\varphi : X_{p+1} \rightarrow Y_{p+1}$ for the vector bundle $\mathcal{S} = \text{Syz}(u^2, v^2, w^2)(3)$ on the Fermat curve Y_{p+1} given by the equation $u^{p+1} + v^{p+1} - w^{p+1}$ over a field of positive characteristic $p \geq 3$. Such a cover corresponds to a representation of the étale fundamental group of Y_{p+1} .

A *Frobenius periodicity* of a vector bundle \mathcal{S} is an isomorphism $F^{e*}\mathcal{S} \rightarrow \mathcal{S}$ for some $e \geq 1$, where F denotes the absolute Frobenius morphism. By a classical result of Lange and Stuhler [8, Satz 1.4], a vector bundle admitting such a Frobenius periodicity is étale trivialisable (the converse does not hold in general though – see [1, example below Theorem 1.1] or [2, Example 2.10]). The proof of Lange and Stuhler yields explicit local equations with gluing data for an étale cover.

Brenner and Kaid showed in [3, Example 5.1] that the bundle \mathcal{S} admits a Frobenius periodicity with $e = 1$. We provide an explicit description of this isomorphism in terms of generators of the syzygy bundle. Then we compute the section ring of X_{p+1} with respect to $\varphi^*\mathcal{O}_{Y_{p+1}}(1)$ and show that the covering obtained via the construction outlined in [8, proof of Satz 1.4] is not geometrically connected. We also compute the genera and the degrees of the connected components. This is a partial answer to a question posed by Brenner and Kaid (see [3, Remark 5.2]).

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1. THE ISOMORPHISM INDUCING FROBENIUS PERIODICITY

Throughout this section we denote $\text{Proj } k[x, y, z]/(x^d + y^d - z^d)$ by C , where k is a field of prime characteristic $p = 2d - 1$. The goal of this section is to explicitly identify the isomorphism $F^*\text{Syz}(x, y, z) \cong \text{Syz}(x, y, z)(-\frac{3(p-1)}{2})$ described in [3, Theorem 3.4], where $F : C \rightarrow C$ is the (absolute) Frobenius morphism. From this isomorphism we will then obtain the desired F -periodicity by passing to a suitable Fermat cover. For a locally free sheaf \mathcal{S} we call a global section of $\mathcal{S}(m)$ a section of *total degree* m .

1.1. Lemma. The locally free sheaf $\mathcal{S}_1 = \text{Syz}(x^p, y^p, x^{\frac{p+1}{2}} + y^{\frac{p+1}{2}})$ on C is generated by the relations

$$R_0 = (y^{\frac{p-1}{2}}, x^{\frac{p-1}{2}}, -(xy)^{\frac{p-1}{2}}) \text{ and } R_1 = (-x, y, x^{\frac{p+1}{2}} - y^{\frac{p+1}{2}})$$

in degree $p+1$ and $\frac{3p-1}{2}$ respectively. And we have an isomorphism $\mathcal{O}_C(-p-1) \oplus \mathcal{O}_C(-\frac{3p-1}{2}) \xrightarrow{R_1, R_0} \mathcal{S}_1$.

Proof. One easily verifies that R_0 and R_1 are indeed syzygies of \mathcal{S}_1 . Moreover, \mathcal{S}_1 is just the pull back of $\text{Syz}_{\mathbb{P}^1}(x^p, y^p, x^{\frac{p+1}{2}} + y^{\frac{p+1}{2}})$ along the Noether normalisation $\pi : C \rightarrow \mathbb{P}^1 = \text{Proj } k[x, y]$ – cf. [3, Remark 2.3].

Note that R_0 and R_1 are linearly independent over $k[x, y]$. Hence, we obtain an exact sequence $0 \rightarrow \mathcal{O}_C(-p-1) \oplus \mathcal{O}_C(-\frac{3p-1}{2}) \rightarrow \mathcal{S}_1 \rightarrow \mathcal{T} \rightarrow 0$, where \mathcal{T} is a torsion sheaf. Taking cohomology we obtain that $H^0(C, \mathcal{T})$ is zero ($H^1(C, \mathcal{T})$ vanishes since \mathcal{T} has support on a closed affine subscheme). Hence, we have the desired isomorphism. \square

1.2. Lemma. The locally free sheaf $\mathcal{S}_2 = \text{Syz}(x^p, y^p, (x^{\frac{p+1}{2}} + y^{\frac{p+1}{2}})^2)$ on C is generated by the relations

$$R_2 = (xy^{\frac{p-1}{2}}, 2x^{\frac{p+1}{2}} + y^{\frac{p+1}{2}}, -y^{\frac{p-1}{2}}) \text{ and } R_3 = (x^{\frac{p+1}{2}} + 2y^{\frac{p+1}{2}}, x^{\frac{p-1}{2}}y, -x^{\frac{p-1}{2}})$$

in degree $\frac{3p+1}{2}$. And we have an isomorphism $\mathcal{O}_C(-\frac{3p+1}{2})^2 \xrightarrow{R_2, R_3} \mathcal{S}_2$.

Proof. The argument is similar to the previous lemma. \square

1.3. Lemma. The kernel of the surjective morphism

$$\mathcal{O}_C^3 \longrightarrow \mathcal{S}_1(p+1) \oplus \mathcal{S}_2(\frac{3p+1}{2}) \xrightarrow{\varphi} \text{Syz}(x^p, y^p, z^p)(\frac{3p+1}{2}),$$

where the first morphism is given by mapping e_i to R_i ($i = 1, 2, 3$) and φ is given by

$$(f_1, f_2, f_3), (g_1, g_2, g_3) \mapsto (z^{d-1}f_1 + g_1, z^{d-1}f_2 + g_2, f_3 + zg_3),$$

is generated by the single relation $(z, -y, x)$.

Proof. By [3, Steps 3 and 4 of Theorem 3.4] we have that the morphism is surjective and that its kernel is isomorphic to $\mathcal{O}_C(-1)$. A straightforward calculation shows that $(z, -y, x)$ is mapped to zero along this map. \square

By abuse of notation we will denote this morphism by φ . With this notation we have

$$\begin{aligned} \varphi(e_1) &= (-z^{d-1}x, yz^{d-1}, x^d - y^d), \\ \varphi(e_2) &= (xy^{d-1}, 2x^d + y^d, -y^{d-1}z), \\ \varphi(e_3) &= (x^d + 2y^d, x^{d-1}y, -x^{d-1}z). \end{aligned}$$

We thus obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_C(-1) \xrightarrow{(z, -y, x)} \mathcal{O}_C^3 \xrightarrow{\varphi} \text{Syz}(x^p, y^p, z^p)(\frac{3p+1}{2}) \longrightarrow 0.$$

1.4. Proposition. The morphism $\alpha : \text{Syz}(x^p, y^p, z^p)(\frac{3p+1}{2}) \rightarrow \text{Syz}(x, y, z)(2)$ given on generators by

$$\begin{aligned} (-z^{d-1}x, yz^{d-1}, x^d - y^d) &\mapsto (-y, x, 0), \\ (xy^{d-1}, 2x^d + y^d, -y^{d-1}z) &\mapsto (-z, 0, x), \\ (x^d + 2y^d, x^{d-1}y, -x^{d-1}z) &\mapsto (0, -z, y) \end{aligned}$$

is an isomorphism.

Proof. The Koszul complex of $\text{Syz}(z, -y, x)$ yields the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-1) \xrightarrow{(z, -y, x)} \mathcal{O}_C^3 \xrightarrow{\psi} \text{Syz}(z, -y, x)(2) \longrightarrow 0,$$

where ψ is given by

$$\begin{aligned} e_1 &\longmapsto (x, 0, -z), \\ e_2 &\longmapsto (y, z, 0), \\ e_3 &\longmapsto (0, x, y). \end{aligned}$$

Mapping (a_1, a_2, a_3) to $(a_3, -a_2, a_1)$ yields an isomorphism $\text{Syz}(z, -y, x)(2) \rightarrow \text{Syz}(x, y, z)(2)$. Together with the observation after Lemma 1.3 this yields the desired isomorphism $\text{Syz}(x^p, y^p, z^p)(\frac{3p+1}{2}) \rightarrow \text{Syz}(x, y, z)(2)$. \square

1.5. Corollary. Let k be a field of characteristic $p \geq 3$ and let

$$Y_{p+1} = \text{Proj } k[u, v, w]/(u^{p+1} + v^{p+1} - w^{p+1}).$$

Then we have the Frobenius periodicity $F^* \text{Syz}(u^2, v^2, w^2)(3) \rightarrow \text{Syz}(u^2, v^2, w^2)(3)$ given on generators by

$$\begin{aligned} (-w^{p-1}u^2, v^2w^{p-1}, u^{p+1} - v^{p+1}) &\longmapsto (-v^2, u^2, 0), \\ (u^2v^{p-1}, 2u^{p+1} + v^{p+1}, -v^{p-1}w^2) &\longmapsto (-w^2, 0, u^2), \\ (u^{p+1} + 2v^{p+1}, u^{p-1}v^2, -u^{p-1}w^2) &\longmapsto (0, -w^2, v^2). \end{aligned}$$

Proof. Set $d = \frac{p+1}{2}$. Then the isomorphism is induced by the finite cover $C^{2d} \rightarrow C$ given by $x \mapsto u^2, y \mapsto v^2, z \mapsto w^2$ (cf. [3, Example 5.1]) and the isomorphism described in Proposition 1.4. \square

1.6. Remark. Brenner and Kaid actually established a Frobenius periodicity for $\text{Syz}(u^2, v^2, w^2)(3)$ on curves $C = \text{Proj } k[u, v, w]/(u^{2d} + v^{2d} - w^{2d})$, where k is a field of characteristic $p \equiv -1 \pmod{2d}$. Let us write $p = d(l+1) - 1$ with l odd. It would be even more interesting to have explicit descriptions of the isomorphisms $F^* \text{Syz}(u^2, v^2, w^2)(3) \rightarrow \text{Syz}(u^2, v^2, w^2)(3)$ in the case where d is fixed and l varies. For then one would have a relative curve over $\text{Spec } \mathbb{Z}$ which might be interesting with respect to the Grothendieck-Katz p -curvature conjecture – cf. [3, Remark 5.3].

2. SOME COMPUTATIONS

Throughout this section k is a field of characteristic $p = 2d - 1$ and $Y = \text{Proj } k[u, v, w]/(u^{2d} + v^{2d} - w^{2d})$. We denote the Frobenius periodicity described in Corollary 1.5 by $\alpha : F^* \text{Syz}(u^2, v^2, w^2)(3) \rightarrow \text{Syz}(u^2, v^2, w^2)(3)$. We write $\mathcal{S} = \text{Syz}(u^2, v^2, w^2)(3)$. The locally free sheaf \mathcal{S} is generated by

$$\begin{aligned} s_1 &= (-v^2, u^2, 0), \\ s_2 &= (-w^2, 0, u^2), \\ s_3 &= (0, -w^2, v^2) \end{aligned}$$

in total degree 1. Furthermore, we write $\mathcal{S}' = F^* \mathcal{S} = \text{Syz}(u^{2p}, v^{2p}, w^{2p})(3p)$, for which we fix generators

$$\begin{aligned} s'_1 &= (-w^{2d-2}u^2, v^2w^{2d-2}, u^{2d} - v^{2d}), \\ s'_2 &= (u^2v^{2d-2}, 2u^{2d} + v^{2d}, -v^{2d-2}w^2), \\ s'_3 &= (u^{2d} + 2v^{2d}, u^{2d-2}v^2, -u^{2d-2}w^2) \end{aligned}$$

in total degree 1.

The goal of this section is to collect the data that are needed for construction of the étale cover as outlined in [8, Satz 1.4]. For the convenience of the reader we shall review this (with notations tailored to our situation).

Assume that we have a smooth curve Y and a locally free sheaf \mathcal{S} on Y with Frobenius periodicity $\alpha : F^*\mathcal{S} \rightarrow \mathcal{S}$. Let U_1, U_2 be a trivialising open cover for \mathcal{S} . Let $\psi_{U_1} : \mathcal{S}|_{U_1} \rightarrow \mathcal{O}_{U_1}^2$ and $\psi_{U_2} : \mathcal{S}|_{U_2} \rightarrow \mathcal{O}_{U_2}^2$ be the transition mappings and denote $\psi_{U_1}\psi_{U_2}^{-1} \in \mathrm{GL}_2(\mathcal{O}_{U_1 \cap U_2}^2)$ by T . For a matrix $A = (a_{ij})$ with coefficients in a ring denote by $A^{(p)}$ the matrix whose entries are the a_{ij}^p . We denote by H_{U_1} the mapping that makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{S}|_{U_1} & \xrightarrow{\alpha^{-1}|_{U_1}} & \mathcal{S}'|_{U_1} \\ \downarrow \psi_{U_1} & & \downarrow \psi_{U_1}^{(p)} \\ \mathcal{O}_{U_1}^2 & \xrightarrow{H_{U_1}} & \mathcal{O}_{U_1}^2 \end{array}$$

and similarly for H_{U_2} .

Let now

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where the entries are indeterminates. With this notation one has by [8, proof of Satz 1.4] that a trivialising étale cover $g : X \rightarrow Y$ for \mathcal{S} on Y is obtained by gluing the algebras

$$A_{U_1} = \mathcal{O}_Y(U_1)[a, b, c, d, (\det A)^{-1}] / ((A^{(p)}A^{-1} - H_{U_1})_{i,j} \mid i, j = 1, 2)$$

and

$$B_{U_2} = \mathcal{O}_Y(U_2)[\alpha, \beta, \gamma, \delta, (\det B)^{-1}] / ((B^{(p)}B^{-1} - H_{U_2})_{i,j} \mid i, j = 1, 2)$$

along the the identifications $(TB)_{ij} = (A)_{ij}$ on $U_1 \cap U_2$. The morphism $g : X \rightarrow Y$ is induced by the inclusions $\mathcal{O}_Y(U_1) \rightarrow A_{U_1}$ and $\mathcal{O}_Y(U_2) \rightarrow B_{U_2}$.

2.1. Lemma. Let $Y = \mathrm{Proj} k[u, v, w] / (u^{2d} + v^{2d} - w^{2d})$ and $\mathcal{S} = \mathrm{Syz}(u^2, v^2, w^2)(3)$. Then $U = D_+(u), W = D_+(w)$ is a trivialising cover for \mathcal{S} with respect to the isomorphisms

$$\begin{aligned} \psi_U : \mathcal{S}|_U &\longrightarrow \mathcal{O}_U^2, & \frac{s_1}{u} &\longmapsto e_1, \frac{s_2}{u} &\longmapsto e_2, \\ \psi_W : \mathcal{S}|_W &\longrightarrow \mathcal{O}_W^2, & \frac{s_2}{w} &\longmapsto e_1, \frac{s_3}{w} &\longmapsto e_2. \end{aligned}$$

The transition mapping for the cover U, W is given by

$$\psi_U \psi_W^{-1} = \begin{pmatrix} 0 & -\frac{w}{v^2} \\ \frac{u}{w} & \frac{u^2}{uw} \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_Y(U \cap W)).$$

Proof. First of all, note that we have the relation $R : w^2 s_1 - v^2 s_2 + u^2 s_3 = 0$ among the fixed generators of \mathcal{S} . On U this relation may be rewritten as $\frac{w^2}{u^2} \frac{s_1}{u} + \frac{s_3}{u} - \frac{v^2}{u^2} \frac{s_2}{u} = 0$ so that $\frac{s_1}{u}, \frac{s_2}{u}$ generate $\mathcal{S}|_U$. Consequently, $\mathcal{S}|_U$ is free. Indeed, \mathcal{S} is of rank 2 and we have the exact sequence $\mathcal{O}_U^2 \rightarrow \mathcal{S}|_U \rightarrow 0$ (the $\mathcal{O}_Y(a)$ are invertible – cf. [7, Proposition II.5.12 (a)]). Hence, the kernel is of rank zero, thus zero since \mathcal{O}_U is torsion-free.

One sees similarly that $\mathcal{S}|_W$ is free, generated by $\frac{s_2}{w}, \frac{s_3}{w}$.

We thus have isomorphisms

$$\begin{aligned} \psi_U : \mathcal{S}|_{U \cap W} &\longrightarrow \mathcal{O}_{U \cap W}^2, & \frac{s_1}{u} &\longmapsto e_1, \frac{s_2}{u} &\longmapsto e_2, \\ \psi_W : \mathcal{S}|_{U \cap W} &\longrightarrow \mathcal{O}_{U \cap W}^2, & \frac{s_2}{w} &\longmapsto e_1, \frac{s_3}{w} &\longmapsto e_2. \end{aligned}$$

And we obtain $\frac{s_2}{w} = \frac{u}{w} \frac{s_2}{u}$ and $\frac{s_3}{w} = \frac{v^2}{uw} \frac{s_2}{u} - \frac{w}{u} \frac{s_1}{u}$ using R restricted to $U \cap W$. Whence the transition matrix. \square

2.2. Lemma. Let $Y = \text{Proj } k[u, v, w]/(u^{2d} + v^{2d} - w^{2d})$, where k is a field of characteristic $p = 2d - 1$ and let $\mathcal{S}' = F^*(\text{Syz}(u^2, v^2, w^2)(3)) = \text{Syz}(u^{2p}, v^{2p}, w^{2p})(3p)$. Then $\frac{s'_1}{u}, \frac{s'_2}{u}$ is a basis of $\mathcal{S}'|_U$ and $\frac{s'_2}{w}, \frac{s'_3}{w}$ is a basis of $\mathcal{S}'|_W$.

Proof. Note that we have a relation $w^2 s'_1 - v^2 s'_2 + u^2 s'_3 = 0$. The syzygies s'_i are of total degree $2d + 2p - 3p = 1$ in \mathcal{S}' . Since $\mathcal{S}' = F^* \mathcal{S}$ we have by Lemma 2.1 that $\mathcal{S}'|_U, \mathcal{S}'|_W$ are free and similarly to the proof of Lemma 2.1 one sees that any two generators have to be free.

On U this relation may be written as $\frac{w^2}{u^2} \frac{s'_1}{u} - \frac{v^2}{u^2} \frac{s'_2}{u} + \frac{s'_3}{u} = 0$ so that $\frac{s'_1}{u}, \frac{s'_2}{u}$ generate $\mathcal{S}'|_U$. Similarly we have on W that $\frac{s'_2}{w}, \frac{s'_3}{w}$ generate $\mathcal{S}'|_W$. \square

2.3. Lemma. The change of basis matrix from $B'_U := \{\frac{s'_1}{u}, \frac{s'_2}{u}\}$ to $B_U^{(p)} := \{\frac{s_1^p}{u^p}, \frac{s_2^p}{u^p}\}$ is given by

$$\begin{pmatrix} \frac{v^2 w^{p-1}}{u^{p+1}} & 2 + \frac{v^{p+1}}{u^{p+1}} \\ 1 - \frac{v^{p+1}}{u^{p+1}} & -\frac{v^{p-1} w^2}{u^{p+1}} \end{pmatrix}.$$

And the change of basis matrix from $B'_W := \{\frac{s'_2}{w}, \frac{s'_3}{w}\}$ to $B_W^{(p)} := \{\frac{s_2^p}{w^p}, \frac{s_3^p}{w^p}\}$ is given by

$$\begin{pmatrix} -\frac{u^2 v^{p-1}}{w^{p+1}} & -\frac{u^{p+1} + 2v^{p+1}}{w^{p+1}} \\ -\frac{2u^{p+1} + v^{p+1}}{w^{p+1}} & -\frac{u^{p-1} v^2}{w^{p+1}} \end{pmatrix}.$$

Proof. We first compute the change of basis matrix on U . Write $\alpha \frac{s_1^p}{u^p} + \beta \frac{s_2^p}{u^p} = \frac{s'_1}{u}$. Looking at the second component shows that $\alpha = \frac{v^2 w^{p-1}}{u^{p+1}}$ and restricting to the third component yields $\beta = (1 - \frac{v^{p+1}}{u^{p+1}})$. Similarly one obtains that $(2 + \frac{v^{p+1}}{u^{p+1}}) \frac{s_1^p}{u^p} - \frac{v^{p-1} w^2}{u^{p+1}} \frac{s_2^p}{u^p} = \frac{s'_2}{u}$.

On W we have

$$-\frac{u^2 v^{p-1}}{w^{p+1}} \frac{s_2^p}{w^p} - \frac{2u^{p+1} + v^{p+1}}{w^{p+1}} \frac{s_3^p}{w^p} = \frac{s'_2}{w}$$

and

$$-\frac{u^{p+1} + 2v^{p+1}}{w^{p+1}} \frac{s_2^p}{w^p} - \frac{u^{p-1} v^2}{w^{p+1}} \frac{s_3^p}{w^p} = \frac{s'_3}{w}.$$

\square

2.4. Proposition. The isomorphisms $H_U : \mathcal{O}_U^2 \rightarrow \mathcal{O}_U^2$ and $H_W : \mathcal{O}_W^2 \rightarrow \mathcal{O}_W^2$ are given by

$$H_U = \begin{pmatrix} \frac{v^2 w^{p-1}}{u^{p+1}} & 2 + \frac{v^{p+1}}{u^{p+1}} \\ 1 - \frac{v^{p+1}}{u^{p+1}} & -\frac{v^{p-1} w^2}{u^{p+1}} \end{pmatrix} \text{ and } H_W = \begin{pmatrix} -\frac{u^2 v^{p-1}}{w^{p+1}} & -\frac{u^{p+1} + 2v^{p+1}}{w^{p+1}} \\ -\frac{2u^{p+1} + v^{p+1}}{w^{p+1}} & -\frac{u^{p-1} v^2}{w^{p+1}} \end{pmatrix}$$

with respect to standard bases.

Proof. In order to obtain H_U , note that $H_U = \psi_U^{(p)} \alpha^{-1}|_U \psi_U^{-1}$. Moreover, ψ_U^{-1} is the identity matrix with respect to the bases $B_U := \{\frac{s_1}{u}, \frac{s_2}{u}\}$ and standard basis. Likewise, $\alpha^{-1}|_U$ is the identity matrix with respect to the bases B_U and $B'_U := \{\frac{s'_1}{u}, \frac{s'_2}{u}\}$, as is $\psi_U^{(p)}$ with respect to $B_U^p := \{\frac{s_1^p}{u^p}, \frac{s_2^p}{u^p}\}$ and standard basis. So H_U with respect to standard basis is none other than the matrix computed in Lemma 2.3.

The claim about H_W follows similarly. \square

3. THE EXAMPLE

In this section we compute the section ring of the étale cover

$$g : X \longrightarrow Y = \operatorname{Proj} k[u, v, w]/(u^{p+1} + v^{p+1} - w^{p+1})$$

that is obtained from the Frobenius periodicity

$$F^*(\operatorname{Syz}(u^2, v^2, w^2)(3)) \cong \operatorname{Syz}(u^2, v^2, w^2)(3)$$

via [8, Satz 1.4]. We also show that X is not geometrically connected and we compute the genera and the degrees of its irreducible components when k contains a $(p-1)$ th root of -2 . In fact, in this case any two irreducible components of X are isomorphic. In the following we will denote the ring $k[u, v, w]/(u^{p+1} + v^{p+1} - w^{p+1})$ by R . Note that this is the section ring induced by $\mathcal{O}_Y(1)$ on Y .

Following the construction outlined at the beginning of section 2 and using Lemma 2.1 we obtain that

$$(3.1) \quad a = -\frac{w}{u}\gamma \quad c = \frac{u}{w}\alpha + \frac{v^2}{uw}\gamma$$

$$(3.2) \quad b = -\frac{w}{u}\delta \quad d = \frac{u}{w}\beta + \frac{v^2}{uw}\delta.$$

Explicitly, one has

$$(3.3) \quad \begin{aligned} A_U &= \mathcal{O}_Y(U)[a, b, c, d, \det A^{-1}]/(\det A^{-1}(a^p d - cb^p) - \frac{v^2}{u^2} \frac{w^{p-1}}{u^{p-1}}, \\ \det A^{-1}(b^p a - a^p b) - (2 + \frac{v^{p+1}}{u^{p+1}}), \det A^{-1}(c^p d - cd^p) - (1 - \frac{v^{p+1}}{u^{p+1}}), \\ \det A^{-1}(d^p a - bc^p) + \frac{w^2}{u^2} \frac{v^{p-1}}{u^{p-1}}). \end{aligned}$$

Our first goal is to compute the section ring $S = \bigoplus_{n \geq 0} H^0(X, g^* \mathcal{O}_Y(n))$. Note that we have a finite ring extension $R = k[u, v, w]/(u^{p+1} + v^{p+1} - w^{p+1}) \rightarrow S$ (cf. [4, Lemma 3.5] – the injectivity still holds since the morphisms on the stalks are all injective).

3.1. Lemma. The natural maps $R \longrightarrow A_U[u, u^{-1}]$ and

$$A_U \longrightarrow A_U[u, u^{-1}] \longrightarrow A_U[u, u^{-1}, w^{-1}]$$

are all injective.

Proof. It is enough to show that u, w are non-zero divisors in the section ring S induced by $g^* \mathcal{O}_Y(1)$. Indeed, $A_U[u, u^{-1}]$ is isomorphic to S_u (see e.g. [6, 2.2.1]). In particular, the map $A_U \rightarrow A_U[u, u^{-1}]$ is just the natural inclusion.

Let X_i be a connected component of X , we then obtain a finite morphism $X_i \rightarrow Y$ which is dominant. The pull back of $\mathcal{O}_Y(1)$ to X_i induces a section ring S_i and S is isomorphic to $S_1 \times \dots \times S_n$. As $X_i \rightarrow Y$ is dominant $R \rightarrow S_i$ is injective and S_i is integral since X_i is irreducible. So if u were a zero divisor it would map to 0 in some S_i . But this is impossible since the morphism $R \rightarrow S$ is injective. \square

3.2. Proposition. The section ring induced by $g^* \mathcal{O}_Y(1)$ on X is given by

$$S = R[w\alpha, w\beta, w\gamma, w\delta, \alpha \frac{u^2}{w} + \gamma \frac{v^2}{w}, \beta \frac{u^2}{w} + \delta \frac{v^2}{w}, (\alpha\delta - \beta\gamma)^{-1}]$$

viewed as a subring of $A_U[u, u^{-1}, w^{-1}]$ with the identifications of (3.1) and (3.2).

Proof. Denote the section ring by T . Then one has $T_u = A_U[u, u^{-1}]$ and $T_w = B_W[w, w^{-1}]$ (see e.g. [6, 2.2.1]). Since $D_+(u), D_+(w)$ form a covering of Z it is enough to show that $T_u = S_u$ and $T_w = S_w$.

Using (3.1) and (3.2) we see that $A_U[u, u^{-1}]$ is given by

$$\mathcal{O}_Y(U)[w\gamma, w\delta, \frac{u^2}{w}\alpha + \frac{v^2}{w}\gamma, \frac{u^2}{w}\beta + \frac{v^2}{w}\delta, (ad - bc)^{-1}, u, u^{-1}].$$

One readily computes that $ad - bc = \alpha\delta - \beta\gamma$. So it is clear that $A_U[u, u^{-1}] \subseteq S_u$. Moreover, $((\frac{u^2}{w}\alpha + \frac{v^2}{w}\gamma)w^2 - v^2w\gamma)u^{-2} = w\alpha$ and $(\frac{u^2}{w}\beta + \frac{v^2}{w}\delta)w^2 - u^2w\delta)u^{-2} = w\beta$. Hence, both $w\alpha$ and $w\beta$ are contained in $A_U[u, u^{-1}]$. This shows the converse inclusion.

The equality $S_w = B_W[w, w^{-1}]$ is immediate. \square

3.3. Lemma. We have $ad - bc \notin k$ in S . Moreover, $(ad - bc)^{p-1} = -2$.

Proof. Assume to the contrary that $D = ad - bc$ is contained in k . Then necessarily $D \in k^\times$. In particular, we may replace A_U with the isomorphic ring

$$A'_U = \mathcal{O}_Y(U)[a, b, c, d]/(a^p d - cb^p - D \frac{v^2}{u^2} \frac{w^{p-1}}{u^{p-1}}, b^p a - a^p b - D(2 + \frac{v^{p+1}}{u^{p+1}}), \\ c^p d - cd^p - D(1 - \frac{v^{p+1}}{u^{p+1}}), d^p a - bc^p + D \frac{w^2}{u^2} \frac{v^{p-1}}{u^{p-1}})$$

and consider S as a subring of this quotient.

We invert u in S and then kill w , i.e. we look at the ring $T := A'_U[u^{-1}]/(w)$. Thinking of this ring as a quotient of $R[u^{-1}]/(w)[a, b, c, d]$ we can rewrite the defining ideal as follows (note that we have $u^{p+1} + v^{p+1} = 0$),

$$(a^p d - cb^p, b^p a - a^p b - (ad - bc), c^p d - cd^p - 2(ad - bc), d^p a - bc^p).$$

This ring is not the zero ring and we still must have $D \in k^\times$ in T . But this is impossible since the defining ideal is contained in (a, b, c, d) .

Since $\det A^{(p)} = (\det A)^p$ and by the multiplicative property of the determinant we obtain $\det A^{p-1} = (ad - bc)^{p-1} = \det H_U = -2$ in A_U . \square

3.4. Corollary. The étale cover $X \rightarrow Y$ obtained via the Frobenius periodicity $F^*S \rightarrow S$ is not geometrically connected.

Proof. As X is étale the section ring is a direct product of normal domains. Assuming that k is algebraically closed, the section ring is a domain if and only if $H^0(X, \mathcal{O}_X) = k$. As we have seen $ad - bc \in H^0(X, \mathcal{O}_X)$. So if X were connected then $ad - bc \in k$. But this is not the case by Lemma 3.3. \square

3.5. Proposition. Assume that k contains a $(p-1)$ th root of -2 which we call η . Then X has exactly $p-1$ connected components which are all isomorphic. The connected components X_i are isomorphic to $\text{Proj } S/(ad - bc + \zeta^i \eta)$, where ζ is a primitive $(p-1)$ th root of unity¹ and S denotes the section ring associated to $g^*\mathcal{O}_Y(1)$ (i.e. the one described in Proposition 3.2).

Proof. We claim that the irreducible components of X are the $X_i = \text{Proj } S/(ad - bc + \zeta^i \eta)$ for $i = 1, \dots, p-1$. At least one of the X_i is non-empty, for otherwise all the $ad - bc + \zeta^i \eta$ were units and thus their product $(ad - bc)^{p-1} + 2$ would be nonzero – contradicting Lemma 3.3.

Next we will show that the rings $T_i = S/(ad - bc + \zeta^i \eta)$ are all isomorphic. So fix indices i, j . We obtain an automorphism of $A_U[w^{-1}, u, u^{-1}]$ by multiplying a, c by ζ^{-j} and by multiplying b, d with ζ^i . Since the ideal in (3.3) is mapped to itself this is well-defined. Furthermore, this induces an automorphism of S . Finally,

¹In other words, any generator of \mathbb{F}_p^\times .

$(ad-bc+\zeta^i\eta)$ is mapped to $(ad-bc+\zeta^j\eta)$. Hence, this also induces an isomorphism of T_i with T_j . It follows in particular that all the X_i are non-empty.

We still have to show that the X_i are the irreducible components. Since S is normal, we have $S = S_1 \times \dots \times S_n$, where the S_i are normal integral k -domains. As $(ad-bc)^{p-1} = -2$ we must have that $ad-bc \mapsto (\zeta^{i_1}\eta, \zeta^{i_2}\eta, \dots, \zeta^{i_n}\eta)$. Here we see that the T_i are therefore of the form $S_j \times \dots \times S_{j+k}$. In particular, the T_i are again normal. As the zeroth graded component of S is $k[ad-bc]$ we obtain that the zeroth graded component of T_i is k . Hence, the T_i are integral domains. We therefore also obtain $n = p-1$ and the X_i have to be the irreducible components. \square

3.6. Lemma. Let R be a commutative ring and let A, B, C square matrices of dimension n with entries in R . Assume furthermore that $\det B$ is invertible in R . Then the ideal generated by the entries of $G := (AB^{-1} - C)$ is equal to the ideal generated by the entries of $H := (A - CB)$

Proof. Write $G = (g_{ij})$ and similarly for B, H . Multiplying G with B from the right we obtain $\sum_j g_{ij}b_{jl} = h_{il}$. This proves one inclusion. The other inclusion is obtained by multiplying H from the right by $B^{-1} = (\det B)^{-1}B^\#$, where $B^\#$ denotes the adjoint matrix. \square

3.7. Proposition. Assume that k contains a $(p-1)$ th root of -2 and let $X_i = \text{Proj } S/(r_i)$ be an irreducible component of $X = \text{Proj } S$. Then the degree of the induced morphism $g_i : X_i \rightarrow Y = \text{Proj } k[u, v, w]/(u^{2d} + v^{2d} - w^{2d})$ is $p(p^2-1)$.

Proof. The degree of the morphism is the k -dimension of the global sections of a fiber. Fix the point $u, w = 1, v = 0$ on Y . Then the matrix H_U in Proposition 2.4 reduces to

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

So by Lemma 3.6 we obtain that the fiber (of the whole covering) is given by modding out $(2c - a^p, 2d - b^p, a - c^p, b - d^p, (ad-bc)^{p-1} + 2)$ in $k[a, b, c, d]$. Observe, that c and d are units in this quotient ring. Indeed, modding out c we obtain $a = 0$ from the third generator and therefore, using the last generator, $2 = 0$. Hence, c had to be a unit. The proof for d works similarly. We may thus rewrite this ideal as $(c^{p^2-1} - 2, d^{p^2-1} - 2, (cd^p - c^pd)^{p-1} + 2)$.

In counting (\bar{k} -valued) points we see from the first two generators that points (c, d) have to satisfy $d \mapsto \zeta c$ and $c^{p^2-1} = 2$, where ζ is a (p^2-1) th root of unity. Furthermore, we must have $\zeta^{p-1} \neq 1$. Otherwise we would obtain a contradiction from the last equation. We claim that all these remaining choices yield points.

Indeed, we then have

$$\begin{aligned} (cd^p - c^pd)^{p-1} &= (c^{p+1}\zeta^p - c^{p+1}\zeta)^{p-1} = c^{p^2-1}\zeta^{p-1}(\zeta^{p-1} - 1)^{p-1} \\ &= 2\zeta^{p-1} \frac{\zeta^{p(p-1)} - 1}{\zeta^{p-1} - 1} = 2 \frac{\zeta^{(p+1)(p-1)} - \zeta^{p-1}}{\zeta^{p-1} - 1} = -2. \end{aligned}$$

So the dimension of a fiber of the whole covering is $(p^2-1)(p^2-1-(p-1)) = (p^2-1)p(p-1)$. Since any two irreducible components are isomorphic by Proposition 3.5 we obtain the dimension of the fiber over an irreducible component by dividing by the number of irreducible components, which is $p-1$ – whence the claim. \square

Summing up what we have proved we obtain

3.8. Theorem. Let k be a field of characteristic $p = 2d-1$ containing a $(p-1)$ th root of -2 , $R = k[u, v, w]/(u^{p+1} + v^{p+1} - w^{p+1})$, $Y = \text{Proj } R$ and $\mathcal{S} = \text{Syz}(u^2, v^2, w^2)(3)$. Then there is a Frobenius periodicity $F^*(\text{Syz}(u^2, v^2, w^2)(3)) \cong \text{Syz}(u^2, v^2, w^2)(3)$.

The trivialising étale cover $g : X \rightarrow Y$ obtained via this periodicity along the construction outlined in [8, Satz 1.4] is given by

$$\text{Proj } R[w\alpha, w\beta, w\gamma, w\delta, \alpha \frac{u^2}{w} + \gamma \frac{v^2}{w}, \beta \frac{u^2}{w} + \delta \frac{v^2}{w}, (\alpha\delta - \beta\gamma)^{-1}]$$

viewed as a subring of $A_U[u, u^{-1}, w^{-1}]$, where

$$\begin{aligned} A_U &= \mathcal{O}_Y(D_+(u))[a, b, c, d, \det A^{-1}] / (\det A^{-1}(a^p d - cb^p) - \frac{v^2}{u^2} \frac{w^{p-1}}{u^{p-1}}, \\ &\det A^{-1}(b^p a - a^p b) - (2 + \frac{v^{p+1}}{u^{p+1}}), \det A^{-1}(c^p d - cd^p) - (1 - \frac{v^{p+1}}{u^{p+1}}), \\ &\det A^{-1}(d^p a - bc^p) + \frac{w^2}{u^2} \frac{v^{p-1}}{u^{p-1}}) \end{aligned}$$

and

$$\begin{aligned} a &= -\frac{w}{u}\gamma & c &= \frac{u}{w}\alpha + \frac{v^2}{uw}\gamma \\ b &= -\frac{w}{u}\delta & d &= \frac{u}{w}\beta + \frac{v^2}{uw}\delta. \end{aligned}$$

This cover consists of $p-1$ irreducible components X_i which are all isomorphic. The genus of an irreducible component is $p(p^2-1)(\frac{p(p-1)}{2}-1)+1$ and the degree of $g_i = g|_{X_i} : X_i \rightarrow Y$ is $p(p^2-1)$.

Proof. The claim about the genus follows from Hurwitz' theorem ([7, Corollary IV.2.4]) since g is unramified, because the genus of Y is $\frac{p(p-1)}{2}$ and since $\deg g_i = p(p^2-1)$ by Proposition 3.7. \square

3.9. Example. For $p=3$ the bundle $\text{Syz}(u^2, u^2, u^2)(3)$ is already trivial on the Fermat quartic $Y = \text{Proj } k[u, v, w]/(u^4 + v^4 - w^4)$ since it is the twisted pull back of a bundle over the quadric $u^2 + v^2 - w^2$. As the quadric is isomorphic to \mathbb{P}_k^1 the bundle splits.

For $p=5$ we consider $\mathcal{S} = \text{Syz}(u^2, v^2, w^2)(3)$ on $Y = \text{Proj } k[u, v, w]/(u^6 + v^6 - w^6)$, where k is a field of characteristic 5 containing a 4th root of -2 . Then \mathcal{S} has no global sections and is thus not the trivial bundle. By Proposition 3.5 the trivialising étale cover $X \rightarrow Y$ has 4 connected components. Each component X_1, X_2, X_3, X_4 is a curve of genus 1261 and the morphisms $X_i \rightarrow Y$ have degree 120.

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